Fermion generations on a co-dimension 2 brane

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We examine the behavior of fermions in the presence of a non-singular thick brane having co-dimension 2. It is shown that one can obtain three trapped zero modes which differ from each other by having different values of angular momentum with respect to the 2 extra dimensions. These three zero modes are located at different points in the extra dimensional space and are interpreted as the three generations of fundamental fermions. The angular momentum in the extra dimensions (which is not conserved) acts as the family or generation label. This gives a higher dimensional picture for the family puzzle.

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I. INTRODUCTION

The brane world idea [1, 2, 3, 4] provides an novel framework for the unification of gauge fields with general relativity and leads to possible new solutions to old problems in particle physics which are not resolved within the Standard Model: the smallness of the cosmological constant, the hierarchy problem, the nature of flavor, the hierarchy of fermion masses and mixings, *etc.*

A key requirement for realizing the brane world idea is that the various bulk fields be localized on the brane. The brane solutions and different matter localization mechanisms have been widely investigated in the scientific literature [5, 6]. It is especially economical to consider models with a purely gravitational localization mechanism, since gravity has the unique feature of having a universal coupling with all matter fields.

Recently the gravitational trapping of zero modes of all bulk fields was demonstrated for the brane solutions with an increasing warp factor [7, 8, 9, 10]. In the present paper, using the solution of [9], we show that three 4-dimensional fermion generations naturally arise on the brane. This gives a purely gravitational mechanism for the origin of three generations of the Standard Model fermions from one generation in a higher-dimensional theory. The localized fermions are stuck at different points in the extra space similar to the model [11].

The main point of this paper is to investigate the properties of higher dimensional fermions in a 2-dimensional curved extra space. In the particular background used it is shown that three fermion zero modes occur. These three zero modes are taken as a model for three different families of fermions. In the present paper we will not address the mass hierarchy or the CKM mixing between different families.

We consider a (1+5)-dimensional bulk space-time with the brane taken as a 4-dimensional string with a 2-dimensional extra space. This solution can be considered as a higher dimensional version of the cosmic string [12] solution. In 2-dimensional spaces anyonic elementary particles, whose angular momentum is not restricted to be integer or half integer, are possible [13]. Whereas in three and higher dimensions all particles must either be integer spin bosons or half integer spin fermions, in two dimensions particles can have any fractional spin and obey fractional statistics. This is because the rotation group in two dimensions is $U(1)$, whose representation is characterized by an arbitrary real (but not necessarily integer) number. A well-known field theoretical example of the anyon is a charge-flux composite state in (2+1)-electrodynamics [14]. The anyons imply the existence of the new physics in the 2-dimensional space. For example, the anyons are expected to play a crucial role in the theory of the fractional quantum Hall effect [15] and the behavior of particles in 2-dimensional materials in condensed matter [16]. Another example is the gravitational anyon (similar to the case considered here), which arises in 2-dimensional spaces with non-trivial topology [17]. This non-trivial topology gives rise to a gravita-

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tional Aharonov-Bohm effect [18]. A similar effect occurs for the brane solution considered in this paper and it gives the condition determining the angular part of the higher dimensional fermion wave function.

II. NON-SINGULAR BRANE SOLUTION

The action we consider in this article is that of gravity in six dimensions

$$S = \int d^6x\sqrt{-g}\left[\frac{M^4}{2}R + \Lambda + L\right] , \qquad (1)$$

where M, R, Λ and L are respectively the fundamental scale, the scalar curvature, the cosmological constant and the Lagrangian of matter fields. All of these physical quantities refer to 6-dimensional space-time with the signature $(+ - - - - -)$.

Variation of the action (1) with respect to the 6-dimensional metric tensor g_{AB} leads to Einstein's equations:

$$R_{AB} - \frac{1}{2}g_{AB}R = \frac{1}{M^4}\left(g_{AB}\Lambda + T_{AB}\right) , \qquad (2)$$

where R_{AB} and T_{AB} are the Ricci and the energy-momentum tensors respectively. Capital Latin indices run over $A, B, ... = 0, 1, 2, 3, 5, 6$.

The two extra dimensions are parameterized by a radial, r, and an angular, θ coordinate. We take the brane to be located at $r = 0$ and assume that rotational invariance is preserved. The brane can be visualized as a 4-dimensional string in six dimensions. In this article for the metric in the bulk space-time we consider the following cylindrically symmetric *ansatz* [7, 8, 9]

$$ds^2 = \phi^2(r)\, ds^2_{(4)} - \lambda(r)\left(dr^2 + r^2 d\theta^2\right) . \qquad (3)$$

Here

$$ds^2_{(4)} = g_{\alpha\beta}\left(x^\nu\right) dx^\alpha dx^\beta \qquad (4)$$

is the metric of 4-dimensional space, where the Greek indices $(\alpha, \beta, ... = 0, 1, 2, 3)$ refer to four dimensions. The function $\lambda(r)$ in (3) must be positive to have space-like extra dimensions.

The 2-dimensional space transverse to the brane usually is described by a conical geometry which has a deficit angle. Because of this deficit angle one can find nonsingular solution of Einstein equations only if the energy-momentum of the brane is proportional to its induced metric [19]. Thus for the stress-energy tensor T_{AB} for the brane we take the cylindrically symmetric in the form [7, 8, 9]

$$T_{\mu\nu} = -g_{\mu\nu}E(r), \quad T_{ij} = -g_{ij}P(r), \quad T_{i\mu} = 0 , \qquad (5)$$

where small Latin indices numerate extra coordinates $(i, j = 5, 6)$. In this paper we do not consider the influence that matter localized on the brane has on the background geometry. Using the *ansatz* (3), the energy-momentum conservation equation gives the relationship between source functions

$$P' + 4\frac{\phi'}{\phi}(P - E) = 0 , \qquad (6)$$

where the prime $= \partial/\partial r$.

To solve the equations (2) we require that the 4-dimensional Einstein equations have the ordinary form without a cosmological term

$$R^{(4)}_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R^{(4)} = 0 . \qquad (7)$$

Then with the *ansätze* (3) and (5) the Einstein field equations (2) become

$$\begin{aligned}
&3\frac{\phi''}{\phi} + 3\frac{\phi'}{r\phi} + 3\frac{(\phi')^2}{\phi^2} + \frac{1}{2}\frac{\lambda''}{\lambda} - \frac{1}{2}\frac{(\lambda')^2}{\lambda^2} + \frac{1}{2}\frac{\lambda'}{r\lambda} = \\
&\qquad\qquad = \frac{\lambda}{M^4}[E(r) - \Lambda] , \\
&\frac{\phi'\lambda'}{\phi\lambda} + 2\frac{\phi'}{r\phi} + 3\frac{(\phi')^2}{\phi^2} = \frac{\lambda}{2M^4}[P(r) - \Lambda] , \qquad (8)\\
&2\frac{\phi''}{\phi} - \frac{\phi'\lambda'}{\phi\lambda} + 3\frac{(\phi')^2}{\phi^2} = \frac{\lambda}{2M^4}[P(r) - \Lambda] .
\end{aligned}$$

These equations are for the $\alpha\alpha$, rr, and $\theta\theta$ components respectively.

Subtracting the rr from the $\theta\theta$ equation and multiplying by ϕ/ϕ' we arrive at

$$\frac{\phi''}{\phi'} - \frac{\lambda'}{\lambda} - \frac{1}{r} = 0 . \qquad (9)$$

This equation has the solution

$$\lambda(r) = \frac{\rho^2\phi'}{r} , \qquad (10)$$

where ρ is an integration constant with units of length.

System (8), after the insertion of (10) and using (6), reduces to only one independent equation. So, taking either the rr or $\theta\theta$ component as independent equation, we have

$$\frac{\phi''}{\phi} + \frac{\phi'}{r\phi} + 3\frac{(\phi')^2}{\phi^2} = \frac{\rho^2\phi'}{2rM^4}\left[P(r) - \Lambda\right] . \qquad (11)$$

In [9] a non-singular solution to (11) with realistic source functions was found of the form

$$\phi = \frac{1 + az^2}{1 + z^2} , \qquad \lambda = \frac{1}{(1 + z^2)^2} , \qquad (12)$$

where $a > 1$ is an integration constant (physically it is the value of the warp factor ϕ at infinity) and

$$z = \frac{r}{\epsilon} , \qquad \epsilon^2 = \frac{40M^4}{\Lambda} \qquad (13)$$

represent a dimensionless radial coordinate and the width of the brane respectively. This solution was constructed with the following boundary conditions

$$\phi(0) = \lambda(0) = 1 \ , \quad \phi|_{r\to\infty} = a \ , \quad \phi'|_{r\to\infty} = 0 \ , \tag{14}$$

where $a > 1$. These conditions mean that at the origin (3) has the form of a 6-dimensional Minkowski metric. Any other values of $\phi(0)$ and $\lambda(0)$ simply correspond to a re-scaling of the coordinates. Using the boundary conditions (14) at $r = 0$ the constant ρ in (10) can be expressed by the constants entering the solution (12)

$$\rho^2 = \frac{\epsilon^2}{2(a-1)} = \frac{20M^4}{(a-1)\Lambda} \ . \tag{15}$$

The solution (12) corresponds to the following choice of the source functions (5)

$$P(r) = \Lambda\left[\frac{4(a+1)}{5\phi} - \frac{3a}{5\phi^2}\right] \ , \quad E(r) = \Lambda\left[\frac{3(a+1)}{5\phi} - \frac{3a}{10\phi^2}\right] \ . \tag{16}$$

At the origin $r = 0$, where $\phi(0) = 1$, the source functions (16) have the value

$$P(0) = \frac{a+4}{5}\Lambda \ , \qquad E(0) = \frac{a+2}{5}\Lambda \ . \tag{17}$$

In the asymptotic region $\phi\,(r)|_{r\gg\epsilon} \to a > 1$ and source functions decrease to

$$P\,(r)|_{r\gg\epsilon} = \frac{4a+1}{5a}\Lambda < P(0) \ , \quad E\,(r)|_{r\gg\varepsilon} = \frac{3a+1}{5a}\Lambda < E(0) \ . \tag{18}$$

We want to point out a problem associated with the source functions. In general the Einstein equations have an infinite number of solutions generated by different matter energy-momentum tensors, most of which have no clear physical meaning. There is a great freedom in the choice of $E(r)$ and $P(r)$; their form is only restricted by (6). It is not easy to construct realistic source functions from fundamental matter fields so that the brane is a stable, localized object. The source functions $E(r)$ and $P(r)$ given in (16) satisfy some physically reasonable conditions: that are smooth functions which decrease monotonically as one moves away from $r = 0$.

III. DEFICIT ANGLE

The conical geometry of the 2 extra dimensions can have interesting global features which are characterized by a deficit angle as one traverses closed paths in the extra dimensions around the 4D brane. In this section we discuss these features as they are important in determining the angular part of fermions moving in (12).

We found that the metric of 6-dimensional space-time has the form

$$ds^2 = \frac{(1+az^2)^2}{(1+z^2)^2}ds^2_{(4)} - \frac{1}{(1+z^2)^2}\left(dr^2 + r^2 d\theta^2\right) \ , \tag{19}$$

where $z = r/\epsilon$. This solution is similar to the interior metric of a cosmic string in harmonic coordinates [20]. Performing the coordinate transformation

$$z = \tan\left(R/\epsilon\right) \tag{20}$$

this metric takes the form

$$ds^2 = \left[\cos^2\left(\frac{R}{\epsilon}\right) + a\sin^2\left(\frac{R}{\epsilon}\right)\right]^2 ds^2_{(4)} - -dR^2 - \frac{\epsilon^2}{4}\sin^2\left(\frac{2R}{\epsilon}\right)d\theta^2 \ , \tag{21}$$

which is similar to the cosmic string metric found in [12].

We note that in some 2-dimensional space with cylindrical symmetry ($0 \le \theta \le 2\pi$)

$$dl^2 = dR^2 + f^2(R)d\theta^2 \tag{22}$$

the circumference of a circle of radius R is equal to $2\pi f(R)$, where $f(R)$ is the value of effective radial function in the given metric. If the space is flat, we would expect that $2\pi f(R) = 2\pi R$. On the other hand, in cone-like spaces this relation will modified to read

$$2\pi f(R) = 2\pi R\left[1 - \frac{\delta}{2\pi}\right] \ , \tag{23}$$

where δ is the deficit angle. Solving this equation for δ one finds

$$\delta = 2\pi\left[1 - \frac{f(R)}{R}\right] = 2\pi[1 - \Delta(R)] \ , \tag{24}$$

where we have introduced the deficit parameter

$$\Delta(R) = \frac{f(R)}{R} \ . \tag{25}$$

The deficit parameter Δ for our metric (21) has the form

$$\Delta(R) = \frac{\epsilon}{2R}\sin\left(\frac{2R}{\epsilon}\right) \ . \tag{26}$$

From the expression (20) we see that effective radial function R of our space (19) is finite, since $R\,(r) \underset{r\to\infty}{\to} \pi\epsilon/2$.

From (25) we see that close to the origin

$$\Delta\,(r) \underset{r\to 0}{\to} 1 \ , \tag{27}$$

and the deficit angle (24) is zero. However, at the core radius ε the deficit angle is large

$$\Delta(\epsilon) = \frac{2}{\pi} \approx 0.6 \ , \tag{28}$$

thus the space becomes conical. At infinity

$$\Delta(r) \underset{r\to\infty}{\to} 0 \ , \tag{29}$$

the deficit angle (24) is $\delta = 2\pi$, this means that space is closed. This geometry is closely related to that of the 6D metric found by Gibbons and Wiltshire [1], or the conical tear drop metric of reference [21].

The topology of the 2-dimensional space near the brane describes a “distorted cone” or a conical tear drop with the brane situated on its tip. At the origin $r \to 0$ the tip of the cone becomes flat. At infinity $r \to \infty$ the scale function $\lambda \to 0$ and determinant of the extra space becomes zero. This situation is similar to the case of the interior of black hole, in which space is closed upon itself.

IV. LOCALIZATION OF NON-FERMIONIC FIELDS

Localization of the 4-dimensional spin-2 graviton on the brane requires that the integral of the gravitational part of the action integral (1) over r and θ for our solutions (19), to be convergent

$$\begin{aligned} S_g &= \frac{M^4}{2}\int dx^6\sqrt{-g}R = \\ &= \frac{M^4}{2}\int_0^{2\pi} d\theta\int_0^{\infty} dr\ r\phi^2\lambda\int dx^4\sqrt{-g^{(4)}}R^{(4)} = \\ &= \rho^2\pi M^4\int_1^a d\phi\ \phi^2\int dx^4\sqrt{-g^{(4)}}R^{(4)} = \\ &= \frac{\pi M^4\rho^2}{3}\left(a^3-1\right)\int dx^4\sqrt{-g^{(4)}}R^{(4)} \ , \end{aligned} \tag{30}$$

where $R^{(4)}$ and $g^{(4)}$ are respectively the scalar curvature and determinant, in four dimensions. The formula for the effective Planck scale, which is two times the numerical factor in front of the last integral in (30),

$$m_{Pl}^2 = \frac{2\pi M^4\rho^2}{3}\left(a^3-1\right) \approx M^4(a\varepsilon)^2 \ , \tag{31}$$

is similar to those from the “large” extra dimension models [2]. As distinct from these models the effective brane width $(a\varepsilon)$ contains the value of warp factor at infinity $a > 1$. This means that for the same fundamental scale M the actual brane width ϵ in our case is smaller then in [2].

It can be checked that similar to (30) integrals of the Lagrangian for matter fields also are convergent [7, 8, 9]. This means that zero-modes of matter fields are localized on the brane with the non-exponential, increasing gravitational factor ϕ.

Indeed, for the spin-0 fields Φ, if we assume that they are independent of extra coordinates, their 6-dimensional action can be cast in the form

$$\begin{aligned} S_\Phi &= \frac{1}{2}\int d^6x\sqrt{-g}g^{AB}\partial_A\overline{\Phi}\partial_B\Phi = \\ &= \pi\frac{\rho^2}{\kappa_\Phi^2}\int_1^a d\phi\phi^2\int d^4x\sqrt{-g^{(4)}}\partial_\mu\overline{\Phi}^{(4)}\partial^\mu\Phi^{(4)} \ , \end{aligned} \tag{32}$$

where κ_Φ is constant with units of length associated with the 4-dimensional scalar field $\Phi^{(4)}$. The integral over the extra coordinates in (32) is the same as for the spin-2 case above. Thus the integral is finite and the spin-0 fields are localized on the brane.

The 6-dimensional action for the $U(1)$ gauge fields in the case of constant bulk components ($A_i = const, i = 5, 6$) reduces to the 4-dimensional Maxwell action multiplied an integral over the extra coordinates

$$\begin{aligned} S_A &= -\frac{1}{4}\int d^6x\sqrt{-g}g^{AB}g^{MN}F_{AM}F_{BN} = \\ &= -\frac{\pi\rho^2}{2\kappa_A^2}\int_1^a d\phi\int d^4x\sqrt{-g^{(4)}}F_{\mu\nu}F^{\mu\nu} \ , \end{aligned} \tag{33}$$

where κ_A is some constant with units of length. This integral is finite and the gauge fields also are localized on the brane.

V. DIRAC’S EQUATION FOR THE BRANE

We now turn our attention to the problem of the localization of the bulk spin-1/2 fermions on the brane. The action integral for the fermion in a curved background has the form

$$S_\Psi = \int d^6x\sqrt{-g}\overline{\Psi}\ i\ h^B_{\widetilde{A}}\Gamma^{\widetilde{A}}D_B\Psi \ , \tag{34}$$

where D_A denote covariant derivatives, $\Gamma^{\widetilde{A}}$ are the 6-dimensional flat gamma matrices and we have introduced the *sechsbein* $h^{\widetilde{A}}_A$ through the usual definition

$$g_{AB} = h^{\widetilde{A}}_A h^{\widetilde{B}}_B\eta_{\widetilde{A}\widetilde{B}} \ , \tag{35}$$

$\widetilde{A}, \widetilde{B}, \ldots$ are local Lorentz indices.

In six dimensions spinor Ψ has eight components and is equivalent to a pair of 4-dimensional Dirac spinors. In this paper we use the following representation of the flat (8×8) gamma-matrices (for the simplicity we shall drop tildes on the local Lorentz indexes when no confusion will occur)

$$\begin{aligned} \Gamma_\nu &= \begin{pmatrix} \gamma_\nu & 0 \\ 0 & -\gamma_\nu \end{pmatrix} \ , \\ \Gamma_r &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ , \\ \Gamma_\theta &= i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \end{aligned} \tag{36}$$

where 1 denotes 4-dimensional unit matrix and γ_ν, γ_5 are ordinary (4×4) gamma-matrices.

The 6-dimensional massless Dirac equation, which follows from the action (34), has the form

$$\left(h^\mu_{\tilde{B}}\Gamma^{\tilde{B}}D_\mu + h^r_{\tilde{B}}\Gamma^{\tilde{B}}D_r + h^\theta_{\tilde{B}}\Gamma^{\tilde{B}}D_\theta\right)\Psi(x^A) = 0 \ . \quad (37)$$

We shall use the following gauge of *sechsbein* for our background metric (3)

$$h_{\tilde{A}}^{\ B} = \left(\delta^B_{\tilde{\mu}}\frac{1}{\phi},\ \delta^B_{\tilde{r}}\frac{1}{\sqrt{\lambda}},\ \delta^B_{\tilde{\theta}}\frac{1}{r\sqrt{\lambda}}\right) \ . \quad (38)$$

Covariant derivatives of spinor field have the forms

$$\begin{aligned} D_\mu\Psi &= \left(\partial_\mu + \frac{1}{2}\omega_\mu^{\tilde{r}\tilde{\nu}}\Gamma_r\Gamma_\nu\right)\Psi \ , \\ D_r\Psi &= \partial_r\Psi \ , \\ D_\theta\Psi &= \left(\partial_\theta + \frac{1}{2}\omega_\theta^{\tilde{r}\tilde{\theta}}\Gamma_r\Gamma_\theta\right)\Psi \ . \end{aligned} \quad (39)$$

From the definition

$$\begin{aligned} \omega_M^{\widetilde{M}\tilde{N}} &= \frac{1}{2}h^{N\widetilde{M}}\left(\partial_M h_N^{\tilde{N}} - \partial_N h_M^{\tilde{N}}\right) - \\ &-\frac{1}{2}h^{N\tilde{N}}\left(\partial_M h_N^{\widetilde{M}} - \partial_N h_M^{\widetilde{M}}\right) - \\ &-\frac{1}{2}h^{P\widetilde{M}}h^{Q\tilde{N}}\left(\partial_P h_{Q\tilde{R}} - \partial_Q h_{P\tilde{R}}\right)h_M^{\tilde{R}} \end{aligned} \quad (40)$$

the non-vanishing components of the spin-connection for the *sechsbein* (38) can be found

$$\omega_\alpha^{\tilde{r}\tilde{\alpha}} = \delta_\alpha^{\tilde{\alpha}}\frac{\phi'}{\sqrt{\lambda}}, \qquad \omega_\theta^{\tilde{r}\tilde{\theta}} = 1 + r\frac{\lambda'}{2\lambda} \ . \quad (41)$$

Then Dirac's equation (37) takes the form

$$\left[\frac{1}{\phi}\Gamma^\mu\partial_\mu + \frac{1}{\sqrt{\lambda}}\Gamma^r\left(\partial_r + \frac{2\phi'}{\phi} + \frac{1}{2r} + \frac{\lambda'}{4\lambda}\right) + \right. \\ \left. + \frac{1}{r\sqrt{\lambda}}\Gamma^\theta\partial_\theta\right]\Psi(x^A) = 0 \ . \quad (42)$$

VI. ANGULAR DEPENDENCE OF WAVE-FUNCTION

As mentioned in the introduction angular momentum in two dimensions differs fundamentally from angular momentum in higher dimensions. In two dimensions the eigenvalue of the angular momentum operator can take values other than integer or half integer. In higher dimensions the angular momentum algebra is non-commutative, whereas in two dimensions it is a trivial commutative algebra. We have only one generator (Σ below), which obviously commutes with itself. As a result, there is no analogue of the quantization of the angular momentum and the only restriction on eigenvalues of the angular operator is the condition to have a single-valued wave-function.

In topologically non-trivial background there generally exists the problem of how to find single-valued wave-functions. The wave-function Ψ of a particle should be single valued once we perform a full rotation by the bulk angle θ around the brane. This is non-trivial task since geometry described by our solution (19) is cone-like with a variable deficit angle.

To find the θ-dependence of $\Psi(x^A)$ we need to examine the parallel transport of a spinor around the gravitational brane solution of (19). The parallel-transported spinor at some angle θ is given in terms of the spinor at $\theta = 0$ by the integration of the condition

$$h_{\tilde{\theta}}^{\ \theta}D_\theta\Psi = h_{\tilde{\theta}}^{\ \theta}\left(\partial_\theta - \frac{i}{2}\omega\Sigma\right)\Psi = 0 \ , \quad (43)$$

where

$$h_{\tilde{\theta}}^{\ \theta} = \frac{1}{r\sqrt{\lambda}} \ , \qquad \omega = \omega_\theta^{\tilde{r}\tilde{\theta}} = 1 + \frac{r\lambda'}{2\lambda} \ , \quad (44)$$

and we had introduced a (8×8)-matrix which is the generalization of the (2×2) Pauli matrix σ_3.

$$\Sigma = i\Gamma_r\Gamma_\theta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (45)$$

Here 1 denotes 4-dimensional unit matrix.

Notice that for the *ansatz* (3) the θ component of angular momentum operator

$$l_\theta = -ih_{\tilde{\theta}}^{\ \theta}\partial_\theta = -\frac{i}{r\sqrt{\lambda}}\partial_\theta \quad (46)$$

is not conserved quantity because of factor $\lambda(r)$. If one transports a spinor parallel to the circle around the brane it will not return to itself but will undergo a rotation through the angle, which will be different at the different distance r from the brane. So it is not possible in general to have a simple separation of the variables r and θ in the wave-function, that will correspond to a single-valued spinor on the brane.

To separate variables r and θ and find the solution of the condition (43) we consider small regions of the extra space where the deficit parameter (25) and spin connection (44) vary slowly and can be considered as constants. Under these restrictions the solution of condition (43) for closed path around the brane is

$$\Psi(x^A) = \exp\left[i\theta\left(n\Delta + \frac{1}{2}\omega\Sigma\right)\right]\Psi(r, x^\nu) \ , \quad (47)$$

where n is an integer number.

Using the intrinsic version of the metric given in (21) we consider two key regions. The first one is the vicinity of the origin $r \approx 0$ (*i.e.* $R \approx 0$), where the metric is flat

$$h_{\tilde{\theta}}^{\ \theta} = \frac{1}{R} \ , \qquad \Delta(0) = \omega(0) = 1 \ . \quad (48)$$

Note that in this region the exponent in (47) becomes $i\theta\left(n + \frac{1}{2}\Sigma\right)$. The term in parentheses is of the form of an

orbital angular momentum plus spin angular momentum, as is expected near $r = 0$ where the space is flat. The other region we consider is close to the brane surface, for example at $r = \sqrt{3}\epsilon$. At this r the metric (21) describes the geometry of exact cone with

$$h_{\tilde{\theta}}^{\theta} = \frac{1}{R\Delta} \; , \qquad \Delta \approx 0.4 \; , \qquad \omega \approx -0.5 \; . \tag{49}$$

In this region we can also separate the variables r and θ similar to the case of the cosmic string [12, 22].

Note that in spite of the fact that the space near the origin is flat the spin-connection ω is not zero there and a spinor parallel transported in a closed circuit around the brane picks up an overall minus sign, which corresponds to an additional rotation of the spinor about its own axis by 2π. This minus sign is a consequence of our choice of frame $h_{\tilde{A}}^{B}$ and not a physical effect in case of flat space-time. However, for our metric (19) at the distance $r = \epsilon$, the spin connection (44) becomes zero, then changes sign and at the distance $r = \sqrt{3}\epsilon$ takes the value $\omega = -1/2$. This changing of spin connection sign is a physical effect and corresponds to appearance of anyonic state in our model, which can carry fractional angular momentum.

VII. LOCALIZATION OF FERMIONS

The condition for the localization of a field on the brane is that its "wave function" in the extra dimensions be normalizable, or that its action integral over r be convergent. As remarked before the extra space in our model is effectively closed, since $R(r \to \infty) \to \pi\epsilon/2$, so in general, all zero-mode solutions of the bulk fields are normalizable. However their wave functions may spread rather widely in the bulk owing to the lack of an exponential warp factor. Thus, in order not to contradict experimental constraints such as the charge conservation law, we should consider only the modes which are localized inside the brane core $r < \epsilon$. This requires the action integral over r to be convergent close to the brane surface.

We approximate the geometry near the origin $r \approx 0$ by a flat metric and near the brane surface $r \approx \epsilon$ by a conical metric. In these regions we can separate the variables and the single-valued spinor wave-functions have the form (47). Then the Dirac equation (42) takes the form

$$\left[\frac{1}{\phi}\Gamma^{\mu}\partial_{\mu} + \frac{1}{\sqrt{\lambda}}\Gamma^{r}\left(\partial_r + \frac{2\phi'}{\phi} + \frac{1}{2r} + \frac{\lambda'}{4\lambda} - \frac{\omega}{2r}\right) + \right.$$
$$\left. + \frac{in\Delta}{r\sqrt{\lambda}}\Gamma^{\theta}\right]\Psi(r, x^{\nu}) = 0 \; . \tag{50}$$

The solution to this equation, if we completely separate the 4-dimensional and extra coordinates, is given by

$$\Psi(r, x^{\nu}) = \sum_{n} \frac{1}{\phi^2 \lambda^{1/4}} \begin{pmatrix} z^{[(\omega-1)/2+n\Delta]}\psi^n(x^{\mu}) \\ z^{[(\omega-1)/2-n\Delta]}\xi^n(x^{\mu}) \end{pmatrix} , \tag{51}$$

where $\psi^n(x^{\mu})$ and $\xi^n(x^{\mu})$ are a pair of 4-dimensional Dirac spinors and $z = r/\epsilon$ is the dimensionless coordinate. The integer number n in (51) corresponds to different possible values of rotational momentum around the brane possessing rotational symmetry of the *ansatz* (3).

We suppose that 4-dimensional spinors $\psi^n(x^{\nu})$ and $\xi^n(x^{\nu})$ in (51) obey the massless Dirac equations on the brane

$$\gamma^{\mu}\partial_{\mu}\psi^n(x^{\nu}) = \gamma^{\mu}\partial_{\mu}\xi^n(x^{\nu}) = 0 \; . \tag{52}$$

Apart from the massless states $\psi^n(x^{\nu})$ and $\xi^n(x^{\nu})$, the 6-dimensional Dirac equation can have solutions, which correspond to the massive fermions localized on the brane. In the single brane models the masses of the bounded states are typically of order of the energy scale $\sim 1/\epsilon$ at which the brane (viewed as a topological defect in higher-dimensional space-time) exists. This means that these states are very heavy and we do not consider them further here. An interesting supposition is that if the energy scale of the extra dimensions is taken to be of the order of the electro-weak scale (~ 1 TeV) one might envision a scenario where some of the fermions are the mass zero modes above and get their mass from some Higgs-like mechanism, while other fermions are massive modes. This might explain why the mass of the top quark is of the electro-weak scale while all other fermions are light compared to the electro-weak scale.

Using the formula for 6-determinant for our *ansatz* (3)

$$\sqrt{-g} = r\phi^4\lambda\sqrt{-g^{(4)}} \; , \tag{53}$$

the action (34) of the spin-1/2 fields takes the form

$$S_{\Psi} = \frac{2\pi\rho\Delta}{\kappa_{\Psi}^2}\sum_{n}\int d^4x \left(iB_n\overline{\psi^n}\gamma^{\nu}\partial_{\nu}\psi^n + \right.$$
$$\left. + iC_n\overline{\xi^n}\gamma^{\nu}\partial_{\nu}\xi^n\right) \; , \tag{54}$$

where κ_{Ψ} is some constant with the unit of length and the factors B_n and C_n are proportional to extra-dimensional parts of the action integral

$$B_n \sim \int dr\frac{\sqrt{\lambda}}{\phi}z^{\omega+2n\Delta} \; ,$$
$$C_n \sim \int dr\frac{\sqrt{\lambda}}{\phi}z^{\omega-2n\Delta} \; . \tag{55}$$

In the expression (54) as the angular variable the quantity $\theta\Delta$ was used. These factors represent the behavior of the "wave-function" with respect to the extra dimensions. The condition that we take as indicating a particular mode n is localized is for the "wave-function" with respect to the extra dimensions to be peaked. In this case because of θ-depending exponent in (47) after integration by θ the mixing between the modes with the different angular number n vanishes. Thus in the present

approximation, where the Δ and ω are taken as constant in different ranges of r, the different generations do not mix. A less severe approximation would give the mixings between the different generations.

At the origin $r = 0$

$$\Delta = \omega = \phi = \lambda = 1 , \tag{56}$$

and the factors (55) are proportional to

$$\begin{aligned} B_n(0) &\sim \frac{1}{2(1+n)} z^{2(1+n)} , \\ C_n(0) &\sim \frac{1}{2(1-n)} z^{2(1-n)} . \end{aligned} \tag{57}$$

Near the brane surface $r = \sqrt{3}\epsilon$

$$\begin{aligned} \frac{\sqrt{\lambda}}{\phi} &= \frac{1}{1+az^2} \approx \frac{1}{az^2} , \\ \Delta \approx 0.4 , &\quad \omega \approx -0.5 , \end{aligned} \tag{58}$$

and the factors (55) are proportional to

$$\begin{aligned} B_n(\sqrt{3}\epsilon) &\sim \frac{1}{(0.8n-1.5)} z^{(0.8n-1.5)} , \\ C_n(\sqrt{3}\epsilon) &\sim -\frac{1}{(0.8n+1.5)} z^{-(0.8n+1.5)} . \end{aligned} \tag{59}$$

To have localization of modes $\psi^n(x^\nu)$, $\xi^n(x^\nu)$ with some angular number n inside the brane core $r < \epsilon$ we require that the factors B_n, or C_n, should have a maximum near the brane *i.e.* between $r \approx 0$ and $r \approx \epsilon$. This means we require $B_n(0)$, $C_n(0)$ to be increasing functions (*i.e.* $\sim z^b$ with $b > 0$) and $B_n(\sqrt{3}\epsilon)$, $C_n(\sqrt{3}\epsilon)$ to be decreasing functions (*i.e.* $\sim z^{-d}$ with $d > 0$).

For the case of zero mode $n = 0$ both factors B_n and C_n are identical and satisfy the above conditions – near $r = 0$ they are proportional to z^2 and near $r = \sqrt{3}\epsilon$ they are proportional to $z^{-3/2}$. Thus two 4-dimensional spinors $\psi^0(x^\nu)$ and $\xi^0(x^\nu)$ (which are undistinguishable from 4-dimensional point of view) are localized on the brane. For $n = 1$ only the factor B_n satisfies the conditions for a maximum near the brane ($B_n(0) \sim z^4$ and $B_n(\sqrt{3}\epsilon) \sim z^{-0.7}$) which corresponds to localization of $\psi^1(x^\nu)$. For the case $n = -1$ only the C_n satisfies the conditions for a maximum near the brane ($C_n(0) \sim z^4$ and $C_n(\sqrt{3}\epsilon) \sim z^{-0.7}$) which corresponds to localization of $\xi^{-1}(x^\nu)$. So totally we have three generation of fermions localized inside the brane core. This gives a brane world picture for the generation puzzle with the angular momentum.

VIII. DISCUSSION

We have demonstrated that inside the brane core of the co-dimension 2 brane solution there naturally arise three massless fermionic zero modes. These can be interpreted as a model for the three generations of the Standard Model with the angular momentum of the extra dimensions acting as the family number. In order to address the fermion mass hierarchy we would need to have some mass generating mechanism. One possibility would be to introduce a scalar fermion interaction term of the form $\Phi\bar{\Psi}\Psi$. The 4D mass of the fermion is then proportional to the integral of the extra dimensional part of the $\Phi\bar{\Psi}\Psi$ with respect to the extra coordinates r and θ as in [23]. Another open question of the present approach is that the approximation used to separate the θ and r dependence leads to no mixing between the different generations *i.e.* the CKM matrix would be the unit matrix. The mixing between the different zero modes and therefore the CKM matrix could be calculated numerically by taking into account the variation of ω and Δ with respect to r. Both the mass hierarchy and mixing will be considered in a future work.

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